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Some New Classes of Hardy Spaces

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Let $B^p = \{f: \|f\| = \sup_{T \geqslant 1} (1/2T) \int_{-T}^{T} |f|^p)^{1/p} < \infty\}$, $1 < p < \infty$. Then B^p is the dual of a function algebra A^q on R (Beurling). In this paper, we study the harmonic extensions of f in B^p and in A^q, and the corresponding Hardy spaces H_{B^p}, H_{A^q}. It is shown that a parallel theory for L^∞, L^1 and BMO, H^1 can be developed for the above pairs. In particular we prove that for $1 < q \leqslant 2$, $(H_{A^q})^*$ is isomorphic to the Banach space

$$\left\{f \text{ real: } \|f\|_{*,p} = \sup_{T \geqslant 1} \left(\frac{1}{2T} \int_{-T}^{T} |f - m_T f|^p\right)^{1/p} < \infty\right\},$$

where $m_T f = (1/2T) \int_{-T}^{T} f$. We also prove Burkholder, Gundy, and Silverstein's maximal function characterization for the new Hardy space H_{A^q}, $1 < q \leqslant 2$.

1. INTRODUCTION

The limits of the averages $(1/2T) \int_{-T}^{T} |f|^2$, $(1/2T) \int_{-T}^{T} f(x+\tau) \bar{f}(x)\, dx$, where f is a locally integrable function on R, were first used by Bohr, Besicovitch, and Stepanoff in the investigation of almost periodic functions and their discrete spectra [3]. Wiener, in his celebrated memoir of generalized harmonic analysis [18], discovered that the above quantities can be extended to study functions with continuous spectra. He treated such f as sample paths of certain stochastic processes (e.g., white light signals, coin tossing) in his pioneering work of probability, and developed the prediction and filtering theory of stationary processes [19].

For $1 < p < \infty$, let

$$B^p = \left\{f: \|f\| = \sup_{T \geqslant 1} \left(\frac{1}{2T} \int_{-T}^{T} |f|^p\right)^{1/p} < \infty\right\}$$

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and

$$M^p = \left\{ f \colon \|f\| = \overline{\lim_{T \to \infty}} \left(\frac{1}{2T} \int_{-T}^{T} |f|^p \right)^{1/p} < \infty \right\}.$$

These two Banach spaces are the appropriate spaces containing the functions considered by Wiener. They have received much attention recently. In particular, the dualities [2, 7, 12, 14], Fourier transformation [2, 7, 16], multipliers [4, 15], and applications to diffusion equations and the Navier–Stokes equation [1] have been studied by various authors.

By applying the integrated Fourier transformation, an analog of the Plancherel theorem was proved on the spaces B^2 and M^2 [7, 16], and hence such spaces preserve certain properties of L^2. In this paper we will consider harmonic extensions of functions in B^p and the related spaces. It is found that, in contrast to the above, a theory parallel to L^1, L^∞ and H^1, BMO can be developed.

In [2] Beurling showed that B^p, $1 < p < \infty$, is the dual of a certain function algebra A^q, $(1/p) + (1/q) = 1$, on R, which can be continuously embedded into L^1 and L^q. Let H_{B^p}, H_{A^p} be the corresponding Hardy spaces. A locally integrable function f on R is said to have the pth *Central Mean Oscillation*, $1 < p < \infty$, if

$$\sup_{T \geqslant 1} \left(\frac{1}{2T} \int_{-T}^{T} |f - m_T f|^p \right)^{1/p} < \infty, \tag{1.1}$$

where $m_T f = (1/2T) \int_{-T}^{T} f$. We denote this class of functions by CMO^p and the above norm by $\| \cdot \|_{*,p}$. The space CMO^p is similar to the John–Nirenberg BMO [13] by restricting to the intervals centered at 0 only. It contains B^p (by identifying constant functions), and is closed under the Hilbert transformation. Among the other results, we prove that

THEOREM A. *For* $1 < p \leqslant 2$, $(1/p) + (1/q) = 1$, *the dual of* H_{A^p} *is isomorphic to the real* CMO^q.

THEOREM B. *For* $1 < p \leqslant 2$ *and for any real* f, $f + i\tilde{f} \in H_{A^p}$ *if and only if* $f^* \in A^p$, *where* f^* *is the nontangential maximal function of* f.

Theorem A is the Fefferman–Stein duality characterization of H^1 [11] adapted to the new space H_{A^p}, and Theorem B is the corresponding adaption of Burkholder *et al.* to the maximal function characterization of H^p space [5].

The paper is organized as follows. In Section 2, we summarize the known results for B^p, A^p, and their dualities relevant to this development. In Section 3, we prove some elementary theorems of harmonic extensions for functions in B^p and A^p, and define the corresponding Hardy spaces.

We prove two theorems of maximal functions in Section 4.

THEOREM C. *For $1<p<\infty$, there exists $c>0$ such that*

$$\|Mf\|_{B^p} \leqslant c\,\|f\|_{B^p}, \qquad \forall f \in B^p,$$

where Mf is the Hardy–Littlewood maximal function of f.

By applying Theorem C and a duality argument, we have

THEOREM D. *For $1<p<\infty$, there exists $c>0$ such that*

$$\|f^*\|_{A^p} \leqslant c\,\|f\|_{H_{A^p}}, \qquad \forall f \in H_{A^p}.$$

In connection with the Hilbert transformation on B^p, the natural space to be considered is CMO^p defined by (1.1). In Section 5 we prove some basic properties of such space. We also introduce a class of measures called *Central Carleson Measure* (C.C. measure) on the upper half plane R^2_+ and prove

THEOREM E. *$f \in CMO^2$ if and only if $|\nabla u(x, y)|^2\,dx\,dy$ is a C.C. measure on R^2_+, where u is the harmonic extension of f on R^2_+.*

In order to prove Theorems A and B, we bring in a new type of atomic space $H^{a,p}$ in Section 6, and show that for $1<p<\infty$, $(H^{a,p})^*$ is isomorphic to the real part of CMO^q. In Section 7, we prove that for $1<p\leqslant 2$ such atomic space can be identified with H_{A^p} by a technique of Calderón [6] and Wilson [20] and Theorem D. Theorem A, B follow as corollaries.

We do not known whether such atomic decomposition holds for $f \in H_{A^p}$, $2<p<\infty$.

2. PRELIMINARIES

Throughout this paper, we will assume $1<p<\infty$, q satisfies $(1/p)+(1/q)=1$; f will denote a complex valued locally integrable function on R; $\approx$ will mean equivalence of two norms or Banach spaces. Let

$$B^p = \left\{ f\colon \|f\| = \sup_{T \geqslant 1} \left(\frac{1}{2T} \int_{-T}^{T} |f|^p \right)^{1/p} < \infty \right\},$$

and let B^p_0 be the subspace of functions f in B^p such that

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f|^p = 0$$

In [16] it is proved that

PROPOSITION 2.1. *Let $f \in B^p$. Then $\int_{-\infty}^{\infty} |f(x)|^p/(1+x^2)\,dx \leqslant c\,\|f\|^p$ for some $c>0$.*

It is clear that $\|\cdot\|$ is equivalent to another norm $\|*\|'$ on B^p defined by

$$\|f\|' = \sup_{T \geqslant 0} \left(\frac{1}{2T+1} \int_{-T}^{T} |f|^p \right)^{1/p}.$$

Let Ω be the set of bounded, positive, integrable even functions ω which are nonincreasing on R^+ and

$$\omega(0) + \int_{-\infty}^{\infty} \omega(x)\,dx = 1.$$

Let

$$A^p = \left\{ f \colon \|f\| = \inf_{\omega \in \Omega} \left(\int |f|^p \omega^{-(p-1)} \right)^{1/p} < \infty \right\}.$$

It follows that if $p=1$, then $A^p = L^1(R)$.

THEOREM 2.2. (Beurling [2]). (i) *A^p is a Banach algebra contained in $L^1 \cap L^p$.* (ii) *$(A^p)^*$ is isometrically isomorphic to $(B^q, \|\cdot\|')$.*

Moreover, it is also known [7] that

THEOREM 2.3. *$(B_0^q, \|\cdot\|')^*$ is isometrically isomorphic to A^p.*

In [12], Feichtinger introduced another pair of equivalent norms on B^p and A^p as

$$\|f\| = \sup_{k \geqslant 0} \left(2^{-(k/p)} \|f\chi_k\|_p\right),$$

and

$$\|g\| = \sum_{k=0}^{\infty} 2^{k/q} \|g\chi_k\|_p,$$

where $f \in B^p$, $g \in A^p$, and χ_k is the characteristic function of P_k, where

$$P_k = \{x \colon 2^{k-1} < |x| \leqslant 2^k\},\ k \geqslant 1; \qquad P_0 = \{x \colon |x| < 1\}.$$

The isomorphic duality of A^p and B^q under this setting is clear. Feichtinger [12] also obtained an "atomic characterization" of A^p:

THEOREM 2.4. *$f \in A^p$ if and only if f admits a representation $f = \sum_{k=0}^{\infty} f_k$ where $\{f_k\}$ are locally integrable functions with support contained in $[-\rho_k, \rho_k]$ and $\sum_{k=0}^{\infty} \rho_k^{1/q} \|f_k\|_p < \infty$.*

Moreover, if we let

$$\|f\|' = \inf\left\{\sum_{k=0}^{\infty} \rho_k^{1/q} \|f_k\|_p\right\},$$

where the infimum is taken over all decompositions as above, then $\|\cdot\|' \approx \|\cdot\|$ *on* A^p.

Since we are only concerned with equivalent norms, we will, if there is no confusion, use $\|\cdot\|_{A^p}$, $\|\cdot\|_{B^p}$, or just $\|\cdot\|$ without specifying which equivalent norms.

We will also need the following:

PROPOSITION 2.5. *Let f be a nonnegative locally integrable function on R; then there exists* $c > 0$ *such that*

$$\sup_{T\geqslant 1} \frac{1}{2T}\int_{-T}^{T} f \leqslant \sup_{T\geqslant 1} \int_{-\infty}^{\infty} f(x)\frac{T}{x^2+T^2}\,dx \leqslant c \sup_{T\geqslant 1} \frac{1}{2T}\int_{-T}^{T} f.$$

Proof. The first inequality is trivial by noticing that

$$\int_{-T}^{T} f(x)\frac{T}{x^2+T^2}\,dx \geqslant \frac{1}{2T}\int_{-T}^{T} f(x)\,dx.$$

The second inequality follows immediately from the proof of Theorem II in [2].

A more extensive treatment of these inequalities can be found in [7] or [16].

3. HARMONIC EXTENSIONS

Let τ_t be the translation operator defined by $(\tau_t f)(x) = f(x-t)$.

LEMMA 3.1. *The spaces* B^p, B_0^p, A^q *are closed under translations* τ_t, *and* $\|\tau_t\| \leqslant c(1+|t|)^{1/p}$ *for some* $c > 0$ *in the respective spaces.*
Moreover $\lim_{t\to 0} \|\tau_t f - f\| = 0$ *for* $f \in B_0^p$ *and* A^p.

Proof. For $f \in B_0^p$, $T \geqslant 1$, we have

$$\frac{1}{2T}\int_{-T}^{T} |\tau_t f|^p = \frac{1}{2T}\int_{-T-t}^{T-t} |f|^p \leqslant \frac{2(T+|t|)}{2T}\left(\frac{1}{2(T+|t|)}\int_{-T-|t|}^{T+|t|} |f|^p\right).$$

This implies that $\tau_t f \in B_0^p$, and $\|\tau_t\| \leqslant (1+|t|)^{1/p}$. A simple duality argument implies that $\|\tau_t\|$ also satisfies the same estimate on A^q and B^p.

For the second part, it is easy to show that the statement holds for functions with compact support. A density argument of such functions in B_0^p and A^p will yield the result.

Let $P_y(x)=y/\pi(x^2+y^2)$ be the Poisson kernel, and let $u(z)=u_y(x)=P_y*f(x)$, where $z=x+iy$, be the harmonic extension of f on the upper half plane R_+^2.

THEOREM 3.2. *Let $f\in B^p$. Then*

(i) *u_y converges to f nontangentially a.e.;*

(ii) *there exists $c>0$ such that $\|u_y\|\leqslant c\,\|f\|$, $\forall y>0$,*

(iii) *if $f\in B_0^p$, then $\lim_{y\to 0}\|u_y-f\|=0$.*

Proof. (i) Let $\phi(z)=(i-z)/(i+z)$ be the conformal mapping from R_+^2 onto the unit disk D and let $F(e^{i\theta})=f(\phi^{-1}(e^{i\theta}))$. By Proposition 2.1,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|F(e^{i\theta})|^p\,d\theta=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|f(x)|^p}{1+x^2}\,dx\leqslant c\,\|f\|^p<\infty.$$

This implies that $F\in L^p(-\pi,\pi)$ and $P_r*F\to F$ a.e. nontangentially. Hence $u_y\to f$ a.e. nontangentially.

(ii) For any $y>0$, $T\geqslant 1$, we have $c_1, c_2>0$ such that

$$\begin{aligned}\frac{1}{2T}\int_{-T}^{T}|u_y|^p&\leqslant c_1\sup_{h\geqslant 1}\int_{-\infty}^{\infty}|u_y(x)|^p\,P_h(x)\,dx \qquad \text{(by Proposition 2.5)}\\ &\leqslant c_1\sup_{h\geqslant 1}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|f(t)|^p\,P_y(x-t)\,dt\right)P_h(x)\,dx\\ &\leqslant c_1\sup_{h\geqslant 1}\int_{-\infty}^{\infty}|f(t)|^p\,P_{y+h}(t)\,dt\\ &\leqslant c_1c_2\,\|f\|^p \qquad \text{(by Proposition 2.5 again).}\end{aligned}$$

(iii) follows from $\lim_{t\to 0}\|\tau_t f-f\|=0, f\in B_0^p$ (Lemma 3.1), and a standard argument of the Poisson kernel.

THEOREM 3.3. *Let $u(z)=u_y(x)$ be a harmonic function on R_+^2. Then*

$$\sup_{y>0}\|u_y\|_{B^p}<\infty \tag{3.1}$$

if and only if there exists $f\in B^p$ such that $u(z)=P_y*f(x)$.

Proof. The sufficiency follows from Theorem 3.2. To prove the necessity, we use the same technique as above to transform $u(x, y)$ to $U(re^{i\theta})$ on the unit disk D. It follows that

$$\sup_{0<r\leqslant 1} \|U(re^{i\theta})\|_p < \infty,$$

and there exists F such that $U(re^{i\theta}) = P_r * F(\theta)$. By transforming back, we conclude that there exists f, such that $u(z) = P_y * f(x)$, where $f(\phi^{-1}(e^{i\theta})) = F(e^{i\theta})$. To check that $f \in B^p$, we observe that

$$\frac{1}{2T}\int_{-T}^{T} |f|^p \leqslant \varliminf_{y\to\infty} \frac{1}{2T}\int_{-T}^{T} |u_y|^p \leqslant \varliminf_{y\to\infty} \|u_y\|_{B^p}^p < \infty.$$

THEOREM 3.4. *Let $f \in A^p$. Then*

(i) *u_y converges to f nontangentially a.e. and in the A^p-norm,*

(ii) *there exists $c > 0$ independent of f such that $\|u_y\| \leqslant c\, \|f\|$.*

Proof. (i) The nontangential convergence follows from $A^p \subseteq L^1 \cap L^p$. The A^p-norm convergence follows from $\lim_{t\to 0} \|\tau_t f - f\| = 0$ and a standard technique of approximation by the Poisson kernel.

Part (ii) follows from Theorem 3.2 (ii) and

$$\langle P_y * f, g\rangle = \langle f, P_y * g\rangle$$

for $f \in A^p$, $g \in B^q$.

THEOREM 3.5. *Let $u(z) = u_y(x)$, $z = x + iy$, be a harmonic function on R^2_+. Then*

$$\sup_{y>0} \|u_y\|_{A^p} < \infty$$

if and only if there exists an $f \in A^p$ such that $u(z) = P_y * f(x)$.

Proof. Since $\|u_y\|_p \leqslant \|u_y\|_{A^p}$, the condition implies that there exists $f \in L^p$ such that $u(z) = P_y * f(x)$. To show that $f \in A^p$, we need only observe that for $g \in B^p_0$,

$$\left|\int fg\right| \leqslant \varliminf_{y\to 0} \int |u_y g| \leqslant \left(\varliminf_{y\to 0} \|u_y\|_{A^p}\right) \|g\|_{B^q_0} \leqslant c\, \|g\|_{B^q_0}$$

for some $c > 0$.

The sufficiency follows from Theorem 3.4(ii).

We define H_{B^p} to be the class of analytic functions $u(z)$ on R^2_+ such that

$$\|u\|_{H_{B^p}} = \sup_{y>0} \|u_y\|_{B^p} < \infty.$$

Similarly we can define H_{A^p}.

PROPOSITION 3.6. *Let* $f \in B^p$ (or A^p). *Then* f *is almost everywhere the nontangential limit of a* $u \in H_{B^p}$ (H_{A^p}, *respectively*) *if and only if* $u(z) = P_y * f(x)$, $z = x + iy$, *is analytic on* R^2_+.

Moreover $\|u\|_{H_{B^p}} \approx \|f\|_{B^p}$ ($\|u\|_{H_{A^p}} \approx \|f\|_{A^p}$, *respectively*).

Proof. The case B^p follows from Theorems 3.2 and 3.3. The case A^p follows from Theorem 3.4 and 3.5.

4. MAXIMAL FUNCTIONS

Let Mf be the maximal function of f defined by

$$Mf(x) = \sup_{x \in I} \frac{1}{|I|} \int_I |f|,$$

where I is an interval containing x. We will also use $M_T f(x)$ to denote the maximal function of f restricted on $[-T, T]$, i.e.,

$$M_T f(x) = \sup_{x \in I \subseteq [-T, T]} \frac{1}{|I|} \int_I |f|, \qquad x \in [-T, T],$$

and $M_T f(x) = 0$, $x \notin [-T, T]$. Let $\lambda_f(\alpha, T) = |\{x \in [-T, T]: M_T f(x) > \alpha\}|$, where α, $T > 0$.

LEMMA 4.1. *For any* $T \geqslant 1$,

(i) $$\lambda_f(\alpha, T) \leqslant \frac{4}{\alpha} \int_{\{x \in [-T, T]: M_T f > \alpha/2\}} |f| \, dx.$$

(ii) *For* $p > 1$,

$$\frac{1}{2T} \int_{-T}^{T} |M_T f|^p \leqslant \frac{2^{p+1} p}{p-1} \frac{1}{2T} \int_{-T}^{T} |f(x)|^p \, dx.$$

Proof. The proof is the same as the one for maximal functions on R [13, Chap. I, Theorem 4.3].

THEOREM 4.2. *For any* $f \in B^p$, *there exists* $c > 0$. *such that*

$$\|Mf\|_{B^p} \leqslant c \, \|f\|_{B^p}.$$

Proof. For $T \geqslant 1$, let $\Pi_1 = \{I: I \subseteq [-3T, 3T]\}$ and $\Pi_2 = \{J: J \cap (R \backslash [-3T, 3T]) \neq \varnothing\}$, where I, J are intervals, and let

$$N_{3T} f(x) = \sup_{x \in J \in \Pi_2} \left(\frac{1}{|J|} \int_J |f| \right).$$

It is clear that for each x,

$$Mf(x) = \max\{M_{3T}f(x), N_{3T}f(x)\}.$$

By Lemma 4.1(ii), it suffices to show that

$$N_{3T}f(x) \leqslant c\,\|f\|, \qquad x \in [-T, T]. \tag{4.1}$$

For this, we assume without loss of generality, that $J = [-a, b] \in \Pi_2$ with $-a < -3T$. Since $x \in [-T, T]$, we have $-T < b$ and hence $a + 2b \geqslant \max\{b, T\}$,

$$\frac{1}{|J|}\int_J |f| = \frac{1}{b+a}\int_{-a}^{b} |f| \leqslant 2\frac{1}{2(b+a)}\int_{-a}^{a+2b} |f| \leqslant 4\frac{1}{2\alpha}\int_{-\alpha}^{\alpha} |f| \leqslant 4\,\|f\|,$$

where $\alpha = \max\{a, a+2b\}$. This proves (4.1).

For any f such that $u(x, y) = P_y * f$ exists, let

$$f^{*,\alpha}(x) = \sup_{t \in \Gamma_\alpha(x)} |u(t, y)|,$$

where $\Gamma_\alpha(x) = \{z = x + iy\colon |x - t| < \alpha y\}$, $\alpha > 0$, is the nontangential maximal function. It is well known that $f^{*,\alpha}(x) \leqslant cMf(x)$ a.e. [13, Chap. I, Theorem 4.2]. We simply denote $f^* = f^{*,1}$ for $\alpha = 1$.

Let

$$f^+(x) = \sup_{y > 0} |u(x, y)|$$

be the radial maximal function. We will prove that f^+ and $f^{*,\alpha}$ are equivalent in the A^p-norm.

LEMMA 4.3. *Let f and ϕ be nonnegative real valued functions on R. Then for $r > 1$,*

$$\int (Mf(x))^r\, \phi(x)\, dx \leqslant c \int |f(x)|^r (M\phi)(x)\, dx,$$

where c depends only on r.

Proof. It was proved by Fefferman and Stein in [10].

PROPOSITION 4.4. *For any $\alpha > 0$, $\|f^{*,\alpha}\|_{A^p} \approx \|f^+\|_{A^p}$.*

Proof. Without loss of generality, we can assume that $\alpha = 1$ and consider f^* only. Note that

$$f^*(x) \leqslant c\{M[(f^+)^{1/p}](x)\}^p$$

(see [11, p. 170]). Hence

$$\begin{aligned}\|f^*\|_{A^p} &\leqslant \sup\left\{\int f^*\phi \colon \|\phi\|_{B^q}=1\right\}\\ &\leqslant c \sup\left\{\int \{M[(f^+)^{1/p}]\}^p\phi \colon \|\phi\|_{B^p}=1\right\}\\ &\leqslant c' \sup\left\{\int f^+(M\phi)\colon \|\phi\|_{B^q}=1\right\} \qquad \text{(by Lemma 4.3)}\\ &\leqslant c''\, \|f^+\|_{A^p} \qquad \text{(by Theorem 4.2).}\end{aligned}$$

The reverse inequality is trivial.

In the following, we will estimate the norm of $f^* \in H_{A^p}$ as an H^1 analog.

THEOREM 4.5. *Let $f \in H_{A^p}$. Then $\|f^*\|_{A^p} \leqslant c\, \|f\|_{H_{A^p}}$.*

Proof. We will prove

$$\int f^*\phi \leqslant c \int |f|(M\phi) \tag{4.2}$$

for $f \in H_{A^p}$, $\phi \in B^q$, $\phi \geqslant 0$. It follows that

$$\begin{aligned}\|f^*\|_{A^p} &\leqslant \sup\left\{\int f^*|\phi| \colon \|\phi\|_{B^q}=1\right\}\\ &\leqslant c \sup\left\{\int |f|(M\phi)\colon \|\phi\|_{B^q}=1\right\}\\ &\leqslant c'\, \|f\|_{A^p},\end{aligned}$$

and $\|f\|_{A^p}$ is equivalent to $\|f\|_{H_{A^p}}$ (Proposition 3.6).

To prove (4.2), we assume $f \not\equiv 0$ and let $B(z)$ be the Blaschke product formed from zeroes of $f(z)$ and let $g(z)=f(z)/B(z)$. It is clear that $|g(x)| = |f(x)|$, $|f(z)| \leqslant |g(z)|$; moreover, g has no zeroes and hence $\sqrt{g}$ is well defined. Applying Lemma 4.3 to $|\sqrt{g}|$ with $r=2$, we get

$$\int |(\sqrt{g})^*|^2\, \phi \leqslant c \int (M\sqrt{g})^2\, \phi \leqslant c' \int |g| M\phi \tag{4.3}$$

for some c, c' independent of g, ϕ. Note that $\sqrt{g}$ is analytic,

$$(\sqrt{g} * P_y)(x) = (\sqrt{g})(x+iy) = \sqrt{g * P_y(x)}.$$

This implies that

$$g^* = ((\sqrt{g})^2)^* = ((\sqrt{g})^*)^2,$$

and (4.3) becomes

$$\int g^*\phi \leqslant c' \int g(M\phi).$$

Inequality (4.2) thus follows from $f^*(x) \leqslant g^*(x)$, $|f(x)| = |g(x)|$, and the above inequality.

5. The Space CMO^p

A function f on R is said to have a *Central Mean Oscillation* of order p if

$$\|f\|_{*,p} = \sup_{T\geqslant 1} \left(\frac{1}{2T}\int_{-T}^{T} |f - m_T(f)|^p\right)^{1/p} < \infty,$$

where $m_T(f) = (1/2T)\int_{-T}^{T} f$. We use CMO^p to denote this class of functions.

The above definition generalizes the concept of BMO by replacing the arbitrary interval I with $[-T, T]$. Note that for BMO, the John–Nirenberg theorem shows that all the norms defined for $1 \leqslant p < \infty$ are equivalent [13]. This is not the case for CMO^p (Proposition 5.2).

Proposition 5.1. *For* $1 < p < \infty, f \in CMO^p$ *if and only if there exists* α_T, $T \geqslant 1$, *such that*

$$\sup_{T\geqslant 1} \frac{1}{2T}\int_{-T}^{T} |f - \alpha_T|^p < \infty.$$

Proof. The necessity is clear. The sufficiency is implied by the following inequality:

$$\left(\frac{1}{2T}\int_{-T}^{T} |f - m_T(f)|^p\right)^{1/p} \leqslant \left(\frac{1}{2T}\int_{-T}^{T} |f - \alpha_T|^p\right)^{1/p} + |m_T f - \alpha_T|$$

$$\leqslant 2\left(\frac{1}{2T}\int_{-T}^{T} |f - \alpha_T|^p\right)^{1/p}.$$

It follows that by identifying constant functions, $B^p \subseteq CMO^p$, and that the inclusion is proper (e.g., $f(x) = \ln|x|$, then $f \in CMO^p \backslash B^p$). If f is an odd function, then $f \in CMO^p$ implies that $f \in B^p$ (since $m_T f = 0$ for all $T \geqslant 1$).

PROPOSITION 5.2. *By identifying constant functions, CMO^p is a Banach space.*

Moreover, if $1<p_1<p_2<\infty$, then $CMO^{p_2}\subseteq CMO^{p_1}$, and *$CMO^{p_2}$ is not dense in CMO^{p_1}.*

Proof. We prove the last statement only. Let

$$A_k=\{x\in R\colon 2^k\leqslant|x|<2^k+2^{k/2}\},\qquad k=0,1,2,...,$$

and let

$$f(x)=\sum_{k=0}^{\infty}2^{k/2p_1}\chi_{A_k}(x)\operatorname{sgn}(x).$$

Since f is an odd function, $m_T(f)=0$. For $2^k\leqslant T<2^{k+1}$,

$$\frac{1}{2T}\int_{-T}^{T}|f-m_T(f)|^{p_1}\leqslant 2^{-k-1}\sum_{j=0}^{k+1}\int_{A_j}|2^{j/2p_1}|^{p_1}\,dx\leqslant 1,$$

which implies $f\in CMO^{p_1}$.

We will show that the distance of f to CMO^{p_2} is positive, and hence CMO^{p_2} will not be dense in CMO^{p_1}. Suppose this were not true. We assume without loss of generality that there exists an odd function $g\in CMO^{p_2}$ with $\|f-g\|_{*,p_1}\leqslant(1/4)$. Note that by a previous remark, $\|g\|_{Bp_2}<\infty$. Let

$$E_k=\{x\in A_k\colon|g(x)|<2^{(k/2p_1)-2}\}.$$

Then

$$(2^{-(k/2)-2}(3/4)^{p_1}|E_k|)^{1/p_1}\leqslant\left(2^{-k-2}\int_{A_k}|f-g|^{p_1}\right)^{1/p_1}<(1/4),$$

and hence

$$|A_k\backslash E_k|\geqslant(1-(2/3^{p_1}))\,2^{(k/2)+1}.$$

Thus

$$2^{-k-2}\int_{A_k}|g|^{p_2}\geqslant 2^{(k/2)[(p_2/p_1)-1]-2p_2-1}(1-(2/3^{p_1})),$$

which tends to ∞ as $k\to\infty$, i.e., $\|g\|_{B^{p_2}}=\infty$. This contradiction completes the proof.

Let

$$f(iy)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(x)\frac{y}{x^2+y^2}\,dx,$$

and let

$$A_p(f) = \sup_{y \geq 1} \int_{-\infty}^{\infty} |f(x) - f(iy)|^p \, P_y(x) \, dx.$$

As a special case of Theorem 1.1 in [8], we have

THEOREM 5.3. *$f \in CMO^p$ if and only if $A_p(f) < \infty$. In this case, there exists k_1, k_2 such that*

$$k_2 \, \|f\|_{*,p}^p \leqslant A_p(f) \leqslant k_1 \, \|f\|_{*,p}^p.$$

COROLLARY 5.4. *$CMO^p \subseteq L^p(dx/(1+x^2))$.*

Proof. By letting $y = 1$, we have

$$\frac{1}{\pi} \int_{-\infty}^{\infty} |f(x) - f(i)|^p \frac{dx}{1+x^2} \leqslant A_p(f) \leqslant k_1 \, \|f\|_{*,p}^p < \infty.$$

This implies that $(f - f(i))$, and hence $f \in L^p(dx/(1+x^2))$.

For $f \in L^p(dx/(1+x^2))$, $p > 1$, we define the Hilbert transformation Hf of f by

$$\begin{aligned} Hf &= \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{1}{x-t} + \frac{t}{1+t^2} \right) f(t) \, dt \\ &= \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|x-t| > \varepsilon} \left(\frac{1}{x-t} + \frac{t}{1+t^2} \right) f(t) \, dt. \end{aligned}$$

THEOREM 5.5. *If $f \in CMO^p$, then $Hf \in CMO^p$ and $\|Hf\|_{*,p} \leqslant c \, \|f\|_{*,p}$.*

Proof. In view of Theorem 5.3, we will show that there exists $c > 0$ such that for each $y > 0$,

$$\int |Hf(x) - Hf(iy)|^p \, P_y(x) \, dx \leqslant c \int |f(x) - f(iy)|^p \, P_y(x) \, dx.$$

Let $\phi_y(z) = (z - iy)/(z + iy)$ be a conformal mapping from R_+^2 onto D. Let g on D be defined by

$$g(\phi_y(z)) = f(z) - f(iy),$$

and let $\tilde{g}$ be the conjugate of g on D. Then $g(0) = 0$ and $\tilde{g}(\phi_y(x)) = Hf(x) - Hf(iy)$. The above inequality reduces to

$$\int_{\partial D} |\tilde{g}(\theta)|^p \, d\theta \leqslant c \int_{\partial D} |g(\theta)|^p \, d\theta,$$

which holds by the M. Riesz theorem.

COROLLARY 5.6. *If $f \in B^p$, then $Hf \in CMO^p$ and*

$$\|Hf\|_{*,p} \leqslant c \inf\{\|f-\alpha\|_{B^p} : \alpha \text{ is a scalar}\}.$$

We remark that there exists $f \in B^p$ such that $Hf \in CMO^p \setminus B^p$; e.g., let $f = \chi_{[0,\infty)}$. Then $Hf(x) = (1/\pi) \ln |x|$ is the required function.

In the following, we will obtain another equivalent condition for CMO^p, $p=2$. Let λ be a regular Borel measure on R^2_+. λ is called a *Central Carleson Measure* (C.C. measure) if

$$N(\lambda) = \sup_{T \geqslant 1} \frac{1}{2T} \lambda([-T, T] \times [0, T]) < \infty.$$

PROPOSITION 5.7. *Let λ be a regular Borel measure on R^2_+. Then*

$$N(\lambda) \approx \sup_{T \geqslant 1} \iint P_T(z)\, d\lambda(z),$$

where $P_T(z) = T/\pi(x^2 + (y+T)^2)$.

Proof. Let $z = x + iy$. Then for $0 \leqslant |x|, y \leqslant T$,

$$P_T(z) = \frac{1}{\pi T((x/T)^2 + (y/T+1)^2)} \geqslant \frac{1}{5\pi T}.$$

It follows that

$$N(\lambda) \leqslant \frac{2}{5\pi} \sup_{T \geqslant 1} \iint P_T(z)\, d\lambda(z).$$

On the other hand, let $A_0 = [-1, 1] \times [0, 1]$ and

$$A_n = \{(x, y) : 2^n T < |x|, y \leqslant 2^{n+1} T\}.$$

Then

$$\begin{aligned}
\iint P_T(z)\, d\lambda(z) &\leqslant \frac{1}{\pi T} \sum_{n=0}^{\infty} \int_{A_n} \frac{1}{(x/T)^2 + (y/T+1)^2}\, d\lambda(x, y) \\
&\leqslant \frac{1}{2\pi T} \sum_{n=0}^{\infty} \frac{1}{2^{2n}} \lambda(A_n) \\
&\leqslant \frac{2}{\pi} N(\lambda).
\end{aligned}$$

By using the Green's theorem, it is proved in [13, p. 237] that

LEMMA 5.8. *If $g(e^{i\theta}) \in L^1(\partial D)$ such that $g(0)=0$, then*

$$\int_D |\nabla g(w)|^2 (1-|w|^2)\, dw \approx \int_{\partial D} |g(e^{i\theta})|^2\, d\theta.$$

THEOREM 5.9. *$f \in CMO^2$ if and only if $y|\nabla u(x,y)|^2\, dx\, dy$ is a C.C. measure on R^2_+ where $u(x,y) = P_y * f(x)$.*

Proof. Let $\phi_T(z) = (z-iT)/(z+iT)$, and let g be defined on D by $g(\phi_T(z)) = f(z) - f(iT)$ as in Theorem 5.5. Then

$$|f(s)-f(iT)|^2\, P_T(s)\, ds = \frac{1}{2\pi} \int_{\partial D} |g(e^{i\theta})|^2\, d\theta.$$

The first expression is hence equivalent to the left side of Lemma 5.8, which after converting back to the upper half plane is

$$\iint_{R^2_+} |\nabla u|^2 \left(1 - \left|\frac{z-iT}{z+iT}\right|^2\right) dx\, dy.$$

Since

$$1 - \left|\frac{z-iT}{z+iT}\right|^2 = \frac{4Ty}{|z+iT|^2} = 4\pi y\, P_T(z),$$

we conclude that $f \in CMO^2$ if and only if $y|\nabla u|^2\, dx\, dy$ is a C.C. measure by Proposition 5.7.

6. ATOMIC DECOMPOSITION AND DUALITIES

A real integrable function ϕ on R is called an (a, p)-atom, $1 < p < \infty$, if there exists a bounded interval I centered at 0, with $|I| \geqslant 2$ such that (i) supp $\phi \subseteq I$, (ii) $\|\phi\|_{L^p} \leqslant |I|^{-(1/q)}$, (iii) $\int_I \phi(x)\, dx = 0$.

We will use $H^{a,p}$ to denote the class

$$\left\{f : f \text{ real}, f = \sum \lambda_k \phi_k,\ \{\phi_k\} \text{ are } (a,p)\text{-atoms}, \sum |\lambda_k| \leqslant \infty\right\},$$

and let

$$\|f\|_{a,p} = \inf\left\{\sum |\lambda_i| : f = \sum \lambda_i \phi_i \text{ as above}\right\}.$$

Under this norm $H^{a,p}$ is a Banach space. It follows directly from the definition that $\|f\|_{a,p}$ can be expressed as

$$\inf\left\{\sum_{k=0}^{\infty} |I_k|^{1/q} \|f_k\|_{L^p}\right\}, \tag{6.1}$$

where the infimum is taken over all representations $f=\sum_{k=0}^{\infty} f_k$ with $\int f_k=0$ and $\operatorname{supp} f_k \subseteq I_k$, I_k is a bounded interval centered at 0, and $|I_k| \geqslant 2$. Comparing with Feichtinger's decomposition of A^p in Theorem 2.4, we have

$$H^{a,p} \subseteq A^p \qquad \text{and} \qquad \|f\|_{A^p} \leqslant \|f\|_{a,p}.$$

In the next section we will show that for $1<p\leqslant 2$, $\|f\|_{a,p}$ is actually equivalent to $\|f\|_{A^p}+\|\tilde{f}\|_{A^p}$ where $\tilde{f}$ is the conjugate of f, and that $H^{a,p}$ is isomorphic to H_{A^p}.

Let $I_k=[-2^k, 2^k]$, $k=0, 1, 2, \ldots$, and for $f\in H^{a,p}$, let

$$\|f\|'_{a,p}=\inf\left\{\sum |\lambda_k| : f=\sum \lambda_k \phi_k,\ \phi_k \text{ is an } (a,p)\text{-atom, supp } \varphi_k \subseteq I_k\right\}.$$

PROPOSITION 6.1. *For* $f\in H^{a,p}$, $\|f\|_{a,p}=\|f\|'_{a,p}$.

Proof. It follows by definition that $\|f\|_{a,p}\leqslant \|f\|'_{a,p}$. On the other hand let $f=\sum \lambda_k \phi_k$, where ϕ_k are (a,p)-atoms. For each n, let $\{\phi_{n(k)}\}$ be the subfamily of $\{\phi_k\}$ such that $\operatorname{supp} \phi_{n(k)} \subseteq I_n$, but $\operatorname{supp} \phi_{n(k)} \nsubseteq I_{n-1}$. Let $b_n=\sum \lambda_{n(k)}\phi_{n(k)}$. Then

$$\|b_n\|_p \leqslant \sum |\lambda_{n(k)}|\ \|\phi_{n(k)}\|_{L^p} \leqslant |I_n|^{-1/q} \sum |\lambda_{n(k)}|.$$

Let

$$\alpha_n=\left(\sum |\lambda_{n(k)}|\right)^{-1} b_n \qquad \text{and} \qquad \mu_n=\sum |\lambda_{n(k)}|.$$

Then $f=\sum \mu_n \alpha_n$ and

$$\sum_n |\mu_n|=\sum_n \sum_{n(k)} |\lambda_{n(k)}|=\sum_k |\lambda_k|.$$

This implies that $\|f\|'_{a,p}\leqslant \|f\|_{a,p}$.

We will use CMO^q_R to denote the class of real functions in CMO^q.

THEOREM 6.2. *The dual of* $H^{a,p}$ *is isomorphic to* CMO^q_R.

Proof. We will prove the theorem in two parts.

(i) $CMO^q_R \subseteq (H^{a,p})^*$. Let $f\in CMO^q_R$ and put $m_I(f)=(1/2|I|)\int_I f$. For any (a,p)-atom ϕ supported by an interval I, we have

$$\left|\int_I \phi f\right| = \left|\int_I \phi(f-m_I(f))\right| \leqslant \|\phi\|_{L^p}\left(\int_I |f-m_I(f)|^q\right)^{1/q}$$
$$\leqslant \left(\frac{1}{|I|}\int_I |f-m_I(f)|^q\right)^{1/q} \leqslant \|f\|_{*,p}.$$

By passing the inequality to $g=\sum \lambda_i\phi_i \in H^{a,p}$, we have $f\in (H^{a,p})^*$ and $\|f\|_{(H^{a,p})^*} \leqslant \|f\|_{*,q}$.

(ii) $(H^{a,p})^* \subseteq CMO^q_R$. Let $L\in (H^{a,p})^*$ and let I be an interval centered at 0, $|I|\geqslant 2$. Denote

$$L^p_0(I)=\left\{g\in L^p(I)\colon \int_I g=0\right\}.$$

Then $\|g\|_{a,p} \leqslant |I|^{1/q}\,\|g\|_{L^p(I)}$ and hence

$$|L(g)| \leqslant \|L\|_{(H^{a,p})^*}\,\|g\|_{a,p} \leqslant \|L\|\;|I|^{1/q}\,\|g\|_{L^p(I)}.$$

This implies $L\in L^p_0(I)^* = L^q(I)/C$, where C is the space of constant functions on I. Hence there exists $f\in L^q(I)$ such that

$$L(g)=\int_I fg, \qquad g\in L^p_0(I). \tag{6.2}$$

The function f is uniquely determined up to a constant. Let $\{I_n\}_{n=1}^{\infty}$ be an increasing sequence of intervals centered at 0 whose union is R. We can select a sequence $\{f_n\}$ such that $f(x)=f_n(x)$ on I_n satisfies (6.2) for each n, and hence (6.2) holds for any interval I centered at 0.

To show that $f\in CMO^q_R$, we note that for any (a,p)-atom ϕ supported by I,

$$\left|\int_I (f-m_I(f))\,\phi\right| = \left|\int_I f\phi\right| = |L(\phi)| \leqslant \|L\|. \tag{6.3}$$

If g is supported by I and $\|g\|_{L^p}=1$, we define an (a,p)-atom by

$$\phi = 2^{-1}\,|I|^{-1/q}\,(g-m_I(g)).$$

It follows from (6.3) and the fact that $f-m_I(f)$ has mean 0 on I, that

$$|I|^{-1/q}\left|\int_I g(f-m_I(f))\right| = |I|^{-1/q}\left|\int_I (g-m_I(g))(f-m_I(f))\right| \leqslant 2\,\|L\|.$$

By taking the supremum on the left side over all g with $\operatorname{supp} g\subseteq I$ and $\|g\|_{L^p}=1$, we have

$$\left(\frac{1}{|I|}\int_I |f-m_I(f)|^q\right)^{1/q} \leqslant 2\,\|L\|.$$

This implies that $f\in CMO^q_R$ and $\|f\|_{*,q} \leqslant 2\,\|L\|$.

7. THE SPACES $H^{a,p}$ AND $H_{A_R^p}$

For any $f \in A^p$, the conjugate $\tilde{f}$ of f is given by

$$\tilde{f}(x) = \int \frac{f(t)}{x-t}\, dt = \lim_{\varepsilon \to 0} \int_{|t| \geqslant \varepsilon} \frac{f(t)}{x-t}\, dt.$$

It is easy to show that $\tilde{f}$ is not necessary in A^p (e.g., let $f = \chi_{[-1,1]}$. Then $\tilde{f}$ is not in L^1 hence not in A^p). We let $H_{A_R^p}$ be the class of real-valued functions $f \in A^p$ such that $\tilde{f} \in A^p$, and let

$$\|f\|_{H_{A_k^p}} = \|f\|_{A^p} + \|\tilde{f}\|_{A^p}.$$

It follows from the open mapping theorem that

PROPOSITION 7.1. *$H_{A_R^p}$ is isomorphic to H_{A^p}.*

In the following, we will identify $H_{A_R^p}$ with the atomic space $H^{a,p}$ for the cases of $1 < p \leqslant 2$.

LEMMA 7.2. *For $1 < p < \infty$, $H^{a,p} \subseteq H_{A_R^p}$, and $\|f\|_{H_{A_R^p}} \leqslant c\, \|f\|_{a,p}$.*

Proof. We first obseve that if ϕ is supported by an interval $I = [-T, T]$, $T \geqslant 1$, $\int \phi = 0$, and if g has compact support, then

$$\begin{aligned}
\left|\int \tilde{\phi} g\right| &= \left|\int \left(\frac{1}{\pi}\int \frac{\phi(t)}{x-t}\, dt\right) g(x)\, dx\right| \\
&= \left|\int \phi(t) \left(\frac{1}{\pi}\int \frac{g(x)}{t-x}\, dx\right) dt\right| \\
&= \left|\int \phi(t) \left(\frac{1}{\pi}\int \left(\frac{1}{t-x} + \frac{x}{x^2+1}\right) g(x)\, dx\right) dt\right| \qquad \left(\text{by} \int \phi = 0\right) \\
&= \left|\int \phi(t)\, Hg(t)\, dt\right| \\
&= \left|\int \phi(t)(Hg(t) - m_I(Hg))\, dt\right| \qquad \left(\text{by} \int \phi = 0\right) \\
&\leqslant (|I|^{1/q}\, \|\phi\|_{L^p})\, \|Hg\|_{*,q} \\
&\leqslant c(|I|^{1/q}\, \|\phi\|_{L^p})\, \|g\|_{B^q} \qquad \text{(Corollary 5.6).}
\end{aligned}$$

If $f \in H^{a,p}$, $f = \sum_k f_k$ as in (6.1), then $f \in A^p$. Also, the above implies

$$\left| \int \tilde{f} g \right| \leqslant c \left(\sum_k |I_k|^{1/q} \, \|f_k\|_{L^p} \right) \|g\|_{B^q},$$

and hence

$$\left| \int \tilde{f} g \right| \leqslant c \, \|f\|_{a,p} \, \|g\|_{B^q}.$$

Since functions with compact supports are dense in B_0^q, the inequality also holds for $g \in B_0^q$. The duality of B_0^q and A^p (Theorem 2.3) implies that

$$\|\tilde{f}\|_{A^p} \leqslant c \, \|f\|_{a,p},$$

i.e., $\tilde{f} \in A^p$. We conclude that $f \in H_{A^p_R}$.

Let C_p denote the class of real-valued functions on R such that both f and $f^* \in A^p$. It follows from Theorem 4.5 that $H_{A^p_R} \subseteq C_p$. In order to show that $C_p \subseteq H^{a,p}$, $1 < p \leqslant 2$, we will construct an atomic decomposition of $f \in C_p$ similar to the one used by Calderón [6] and Wilson [20].

Let $P_0 = \{x\colon |x| \leqslant 1\}$ and let $P_m = \{x\colon 2^{m-1} < |x| \leqslant 2^m\}$, $m \in N$. For any interval I, let

$$\tilde{I} = \{(x, y) \in R^2_+ : (x - y, x + y) \subseteq I\}$$

be the "tent" region and let

$$\tilde{P}_0 = \{(x, y)\colon |x|, |y| \leqslant 1\},$$
$$\tilde{P}_m = \{(x, y)\colon |x|, |y| \leqslant 2^m\} \backslash \tilde{P}_{m-1}, \qquad m \in N.$$

For $g \in A^p$, let

$$E_k = \{x\colon |g(x)| > 2^k\} = \bigcup_{j=k}^{\infty} I_{kj}, \qquad k \in Z,$$

where $\{I_{k,j}\}$ are disjoint intervals. Define

$$\tilde{E}_k = \bigcup_j \tilde{I}_{k,j}, \qquad T^m_{k,j} = (\tilde{I}_{k,j} \backslash \tilde{E}_{k+1}) \cap \tilde{P}_m \qquad \text{(see Fig. 1).}$$

LEMMA 7.3 *Let $g \in A^p$. Then there exists $c_1, c_2 > 0$ such that*

$$c_1 \sum_{m=0}^{\infty} 2^{m/q} \left(\sum_{k=-\infty}^{\infty} 2^{kp} \, |E_k \cap P_m| \right)^{1/p}$$
$$\leqslant \|g\|_{A^p} \leqslant c_2 \sum_{m=0}^{\infty} 2^{m/q} \left(\sum_{k=-\infty}^{\infty} 2^{kp} \, |E_k \cap P_m| \right)^{1/p}.$$

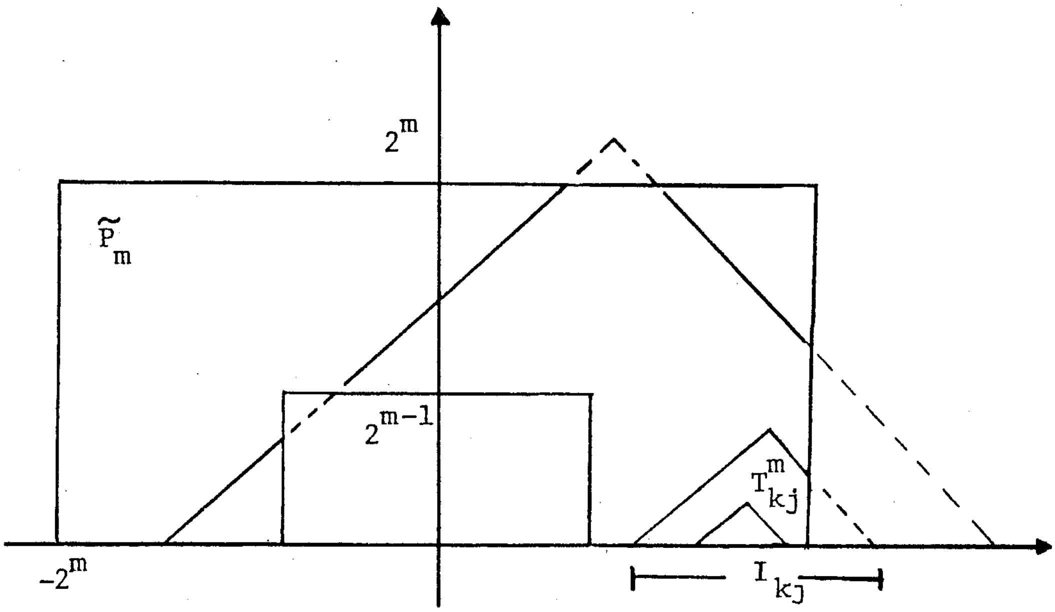

FIGURE 1.

Proof. Note that

$$\begin{aligned}
&\sum_{m=0}^{\infty} 2^{m/q}\left(\sum_{k=-\infty}^{\infty} 2^{kp}\,|E_k\cap P_m|\right)^{1/p}\\
&\quad=\sum_{m=0}^{\infty} 2^{m/q}\left(\sum_{k=-\infty}^{\infty} 2^{kp}\sum_{j=k}^{\infty} |E_j\backslash E_{j+1})\cap P_m|\right)^{1/p}\\
&\quad\leqslant c'\sum_{m=0}^{\infty} 2^{m/q}\left(\sum_{j=-\infty}^{\infty} 2^{jp}\,|(E_j\backslash E_{j+1})\cap P_m|\right)^{1/p}\\
&\quad\leqslant c'\sum_{m=0}^{\infty} 2^{m/q}\,\|g\chi_{p_m}\|_{L^p}.
\end{aligned}$$

The last expression is equivalent to $\|g\|_{A^p}$ (Theorem 2.4).

For the reverse inequality, we have

$$\begin{aligned}
\sum_{m=0}^{\infty} 2^{m/q}\,\|g\chi_{p_m}\|_{L^p}&\leqslant\sum_{m=0}^{\infty} 2^{m/q}\left(\sum_{j=-\infty}^{\infty} 2^{(j+1)p}\,|(E_j\backslash E_{j+1})\cap P_m|\right)^{1/p}\\
&\leqslant 2\sum_{m=0}^{\infty} 2^{m/q}\left(\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{j} 2^{kp}\,|(E_j\backslash E_{j+1})\cap P_m|\right)^{1/p}\\
&=2\sum_{m=0}^{\infty} 2^{m/q}\left(\sum_{k=-\infty}^{\infty} 2^{kp}\sum_{j=k}^{\infty}|(E_j\backslash E_{j+1})\cap P_m|\right)^{1/p}\\
&=2\sum_{m=0}^{\infty} 2^{m/q}\left(\sum_{k=-\infty}^{\infty} 2^{kp}\,|E_k\cap P_m|\right)^{1/p}.
\end{aligned}$$

Let $\phi \in C^\infty(R)$ be a fixed real, even function with $\operatorname{supp} \phi \subseteq \{x: |x| \leqslant 1\}$, $\int \phi = 0$, and $\int_0^\infty e^{-\theta} \hat{\phi}(\theta) = -1$. Let $\phi_y(t) = y^{-1}\phi(t/y)$.

LEMMA 7.4. *For* $1 < p \leqslant 2$, $C_p \subseteq H^{a,p}$.

Proof. We need to show that each $f \in C_p$ admits an (a, p)-atomic decomposition. Let $f \in C_p$ and let $u(x, y) = P_y * f(x)$. Then

$$f(x) = \int_{R_+^2} \frac{\partial u}{\partial y}(t, y)\, \phi_y(x-t)\, dt\, dy \qquad \text{(by [20])}$$

$$= \sum_{m=0}^{\infty} \int_{\bar{P}_m} \frac{\partial u}{\partial y}(t, y)\, \phi_y(x-t)\, dt\, dy.$$

We denote the integral by $f_m(x)$. Note that ϕ_y has compact support in $[-2^m, 2^m]$, hence f_m has compact support in $[-2^{m+1}, 2^{m+1}]$. Also $\int \phi = 0$ implies that $\int f_m = 0$. We claim that

$$\|f_m\|_{L^p} \leqslant \left(\sum_{k=-\infty}^{\infty} 2^{kp}\, |E_k \cap P_m| \right)^{1/p},$$

where $E_k = \{x: f^{*,2} > 2^k\}$. Since $f^* \in A^p$, Proposition 4.4 implies that $f^{*,2} \in A^p$. In view of Lemma 7.3 (by applying $g = f^{*,2}$), $\{f_m\}$ will be the desired decomposition for f.

Let $h \in L^q(R)$, $\|h\|_q = 1$, $(1/p) + (1/q) = 1$. It follows from Holder's inequality that

$$\left| \int h f_m \right| \leqslant \left(\int_{-\infty}^{\infty} \left(\int_0^{\infty} \chi_{\bar{P}_m} y \left| \frac{\partial u}{\partial y} \right|^2 dy \right)^{p/2} dt \right)^{1/p}$$

$$\times \left(\int_{-\infty}^{\infty} \left(\int_0^{\infty} |h * \phi_y(t)|^2 \frac{dy}{y} \right)^{q/2} dt \right)^{1/q}$$

$$\leqslant c_1 \left(\int_{-\infty}^{\infty} \left(\int_0^{\infty} \chi_{\bar{P}_m} y\, |\nabla u|^2\, dy \right)^{p/2} dt \right)^{1/p}$$

(since ϕ is a Littlewood–Paley function, apply Theorem 3.5 in [17, Chapter 7])

$$\leqslant c_2 \left(\sum_{k,j} \int_{-\infty}^{\infty} \left(\int_0^{\infty} \chi_{T_{k,j}^m} y\, |\nabla u|^2\, dy \right)^{p/2} dt \right)^{1/p} \qquad \text{(since } p/2 \leqslant 1\text{)}$$

$$\leqslant c_2 \left[\sum_{k,j} \left(\int_{I_{k,j} \cap P_m} dt \right)^{1-(p/2)} \left(\int_{T_{k,j}^m} y\, |\nabla u|^2\, dy\, dt \right)^{p/2} \right]^{1/p}.$$

By Green's theorem, the double integral is bounded by

$$\int_{\partial T^m_{k,j}} |u|\, y \left|\frac{\partial u}{\partial n}\right| + \frac{1}{2} u^2 \left|\frac{\partial y}{\partial n}\right| ds$$

($\partial/\partial n$ denote the outward normal direction). Both u and $y|\nabla u|$ are bounded by $c_3 2^k$ on $\partial T^m_{k,j}$. Since $|\partial y/\partial n| \leqslant 1$ and $|\partial T^m_{k,j}| \leqslant c_3 |I_{k,j} \cap P_m|$, the integral is bounded by $c_3 2^{2k} |I_{k,j} \cap P_m|$. Note that c_3 is independent of k, j, m, and u; hence the above estimates imply

$$\left|\int hf_m\right| \leqslant c \left(\sum_{k,j} 2^{kp} |I_{k,j} \cap P_m|\right)^{1/p} = c \left(\sum_{k,j} 2^{kp} |E_k \cap P_m|\right)^{1/p}.$$

It follows that $\|f_m\|_{L^p}$ is bounded by the left side, and the claim is proved.

THEOREM 7.5. *For* $1 < p \leqslant 2$ *and for any real* f *on* $R, f + i\tilde{f} \in H_{A^p}$ *if and only if* $f^* \in A^p$.

Proof. Combining Theorem 4.5, Lemma 7.2 and Lemma 7.4, we have

$$H_{A^p_R} \subseteq C_p \subseteq H^{a,p} \subseteq H_{A^p_R}. \tag{7.1}$$

This implies that $f \in H_{A^p_R}$ if and only if $f^* \in A^p$. Hence the theorem follows from Proposition 7.1.

THEOREM 7.6. *For* $1 < p \leqslant 2$, $(H_{A^p})^*$ is isomorphic to CMO^q_R.

Proof. It follows from Theorem 6.3, Proposition 7.1, and that $H_{A^p_R} \approx H^{a,p}$ as in (7.1).

COROLLARY 7.7. *For* $2 \leqslant p < \infty$, $f \in CMO^p$ *if and only if* $f = \Psi_1 + H\Psi_2 + \alpha$ *where* $\Psi_1, \Psi_2 \in B^p$, α *is a constant, and*

$$\|\Psi_1\|_{B^p}, \qquad \|\Psi_2\|_{B^p} \leqslant c\, \|f\|_{*,p}$$

for some constant c.

Moreover, $\|f\|_{*,p} \approx \inf\{\|\Psi_1\|_{B^p} + \|\Psi_2\|_{B^p} : f = \Psi_1 + H\Psi_2 + \alpha\}$.

Proof. The sufficiency follows from Theorem 5.5. To prove the necessity, we note that Theorem 7.6 implies that $f \in (H_{A^p})^* \approx (H_{A^p_R})^*$, and the same argument for the H^1 duality [13, p. 244] can be applied to show that f has the desired representation. The last statement follows similarly as in [13, p. 248].

COROLLARY 7.8. *For* $2 \leqslant p < \infty$, $B^p/H_{B^p} \approx CMO^p_R$.

Proof. For any $\Psi = \Psi_1 + i\Psi_2 \in B^p$, let $\pi\Psi = \Psi_1 + H\Psi_2$. Then π is a bounded linear operator from B^p onto CMO_R^p. Note that $\pi\Psi = 0$ in CMO_R^p if and only if $\Psi_1 + H\Psi_2 = \alpha$, which is equivalent to

$$\Psi_1 + i\Psi_2 = \Psi_1 + i(H\Psi_1 - \alpha) \in H_{B^p}.$$

It follows from the open mapping theorem that B^p/H_{B^p} is isomorphic to CMO_R^p.

COROLLARY 7.9. *For $2 \leqslant p < \infty$, and for $f \in B^p$,*

$$c_2 \|f - iHf\|_{*,p} \leqslant \operatorname{dist}(f, H_{B^p}) \leqslant c_1 \|f - iHf\|_{*,p},$$

where c_1, c_2 are absolute constants.

Proof. Similar to the proof of Corollary 4.6 in [13, Chap. 6].

COROLLARY 7.10. *or $2 \leqslant p < \infty$, and for any real $f \in B^p$, there exist absolute constants c_1, c_2 such that*

$$c_2 \operatorname{dist}(f, H_{B_R^p}) \leqslant \operatorname{dist}_*(Hf, B^p) \leqslant c_1 \operatorname{dist}(f, H_{B_R^p}),$$

where

$$\operatorname{dist}(f, H_{B_R^p}) = \inf\{\|f - \operatorname{Re} g\|_{B^p} : g \in H_{B^p}\},$$

and

$$\operatorname{dist}_*(Hf, B^p) = \inf\{\|Hf - \Psi\|_{*,p} : g \in B^p\}.$$

Proof. Similar to the proof of Corollary 4.7 in [13, Chap. 6].

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Note added in proof. Theorem 7.6 has been extended by José Garcia-Cuerva with $1 < p < \infty$.

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